Neighborhood maps on combinatorial trees and their Markov graphs

Sergiy Kozerenko
(Department of Mathematics, National University of Kyiv-Mohyla Academy, 04070 Kyiv, Ukraine)
E-mail: `kozerenkosergiy@ukr.net`

A combinatorial tree is a finite connected acyclic undirected graph. For a self-map $\sigma : V(X) \to V(X)$ on the vertex set $V(X)$ of a combinatorial tree X, its Markov graph $\Gamma = \Gamma(X, \sigma)$ is defined as a directed graph with the vertex set $V(\Gamma) = E(X)$ and the arc set $A(\Gamma) = \{(u_1v_1, u_2v_2) : u_2, v_2 \in [\sigma(u_1), \sigma(v_1)]_X\}$ (here $[a, b]_X$ denotes the metric interval between a, b in X).

A map $\sigma : V(X) \to V(X)$ on a tree X is a neighborhood map provided $\sigma(u) \in N_G[u]$ for all vertices $u \in V(X)$ (i.e. if $\sigma(u) = u$ or $u\sigma(u) \in E(X)$ for all $u \in V(X)$).

Denote by $d_G(u)$ the vertex degree of u in a graph G and by $L(G)$ the set of all leaf vertices (i.e. vertices u with $d_G(u) = 1$) in G. Also, let $\mathrm{fix}\sigma$ denotes the set of σ-fixed points.

Theorem 1. *For any neighborhood map σ on a tree X with $|V(X)| \geq 2$ the number of weak components in the corresponding Markov graph $\Gamma(X, \sigma)$ equals $\sum_{u \in \mathrm{fix}\sigma} d_X(u) - |\mathrm{fix}\sigma| + 1$.*

Corollary 2. *A neighborhood map σ on a tree X with $|V(X)| \geq 2$ has a weakly connected Markov graph if and only if $\mathrm{fix}\sigma \subset \mathrm{L(X)}$.*

For a map σ on a tree X an edge $uv \in E(X)$ is called σ-positive (σ-negative) provided $d_X(\sigma(u), u) \leq d_X(\sigma(u), v)$ and $d_X(\sigma(v), v) \leq d_X(\sigma(v), u)$ ($d_X(\sigma(u), v) \leq d_X(\sigma(u), u)$ and $d_X(\sigma(v), u) \leq d_X(\sigma(v), v)$). Let $p(X, \sigma)$ and $n(X, \sigma)$ denote the number of σ-positive and σ-negative edges in X, respectively.

Theorem 3. *For any neighborhood map σ on a tree X the number of arcs in the corresponding Markov graph $\Gamma(X, \sigma)$ equals $|E(X)| + 2p(X, \sigma) - \sum_{u \in \mathrm{fix}\sigma} d_X(u)$.*

For a number $\alpha \in \mathbb{R} - \{0, 1\}$ the first general Zagreb index [4] of G is defined as the sum $Z_1^\alpha(G) = \sum_{u \in V(G)} d_G^\alpha(u)$. Similarly, for every number $\alpha \in \mathbb{R} - \{0\}$ the general Randic index [1] of a graph G is the sum $R^\alpha(G) = \sum_{uv \in E(G)} (d_G(u) d_G(v))^\alpha$.

Theorem 4. *For every n-vertex tree X the average number of arcs in Markov graphs for neighborhood maps on X equals*

$$(3 - \frac{2}{n}) \cdot Z_1^{-1}(K_1 + X) + 2R^{-1}(K_1 + X) + \frac{2}{n^2} - \frac{3}{n} - 3.$$

Given a graph G, its Narumi-Katayama index [5] is the product $\mathrm{NK(G)} = \prod_{\mathrm{u \in V(G)}} \mathrm{d_G(u)}$ of degrees over all vertices in G.

Proposition 5. *For every n-vertex tree X the number of its neighborhood maps equals $\frac{1}{n} \cdot \mathrm{NK(K_1 + X)}$.*

References

[1] B. Bollobas and P. Erdos. Graphs of extremal weights. *Ars Combin.*, 50 : 225–233, 1998.
[2] S. Kozerenko. Markov graphs of one-dimensional dynamical systems and their discrete analogues. *Rom. J. Math. Comput. Sci.*, 6 : 16–24, 2016.
[3] S. Kozerenko. Counting the average size of Markov graphs. *J. Int. Math. Virtual Inst.*, 7 : 1–16, 2017.
[4] A. Milićević and S. Nikolić. On variable Zagreb indices. *Croat. Chem. Acta.*, 77 : 97–101, 2004.
[5] H. Narumi and H. Katayama. Simple topological index – A newly devised index characterizing the topological nature of structural isomers of saturated hydrocarbons. *Mem. Fac. Engin. Hokk. Univ.*, 16 : 209–214, 1984.